Recall :

$$\partial_m \Gamma_{ij}^k - \partial_j \Gamma_{im}^k + \Gamma_{ij}^p \Gamma_{mp}^k - \Gamma_{im}^p \Gamma_{jp}^k$$

$$= - II_{im} II_{jp} g^{pk} + II_{ij} II_{mp} g^{pk}$$

L.H.S : depends only on g_{ij} on $\mathcal{U}$ when $X : \mathcal{U} \to M$

And hence invariant under local isometry !!

More about intrinsic geometry :

Recall : $T_pM = \text{span}\{X_u, X_v\}$ varying in $p \in M$.

$\Rightarrow$ $V(u,v) = f^1(u,v) \cdot X_u + f^2(u,v) \cdot X_v$

where $f^1, f^2 \in C^\infty(\mathcal{U})$, belongs to T_pM, $\forall p \in M$.

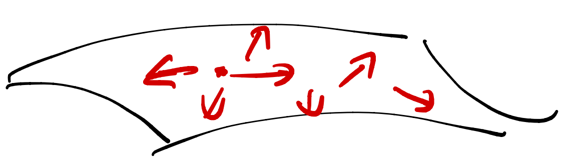
We call this (smooth) vector field on M.

Defn : A vector field W in an open set $V \subseteq M$ of a regular surface M is a correspondence which assign to each $p \in V$ a vector $W(p)$. The vector field is differentiable at $p \in V$ if for some parametrization $X : \mathcal{U} \to M$ around $p \in M$,

$$W(u,v) = \alpha(u,v) \cdot X_u + \beta(u,v) X_v$$ for some

function α, β differentiable at $q = X^{-1}(p) \in \mathcal{U}$.

Picture :



"Hair".

Que: Given a vector field W on M, fixing $p \in M$, and $u \in T_pM$, what is the reasonable variation of W at p along the direction u ??

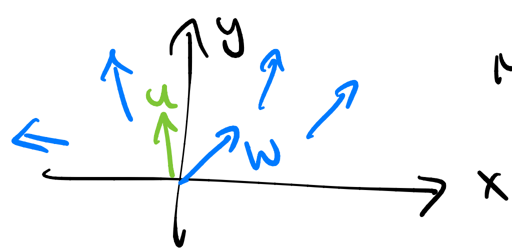
Baby case: $M = \mathbb{R}^2$.

then vector field $W = (W^1, W^2)$ for differentiable fcn $W^1(x,y)$, $W^2(x,y)$ in $(x,y) \in \mathbb{R}^2$.

At p = origin, $0 \in \mathbb{R}^2$, $u \in T_pM \cong \mathbb{R}^2$.

Variation along u at p = classical Directional derivatives

$$= \left(\langle \nabla W^1, u \rangle, \langle \nabla W^2, u \rangle \right) \in \mathbb{R}^2.$$



$M = \mathbb{R}^2$

$$= \lim_{t \to 0} \frac{W(tu) - W(0)}{t} \in \mathbb{R}^2. \quad \text{——} \ (*)$$

We denote this as $\boxed{D_u W|_p}$.

What if $M \neq \mathbb{R}^2$?? How to define (make sense of) $D_u W|_p$??

From $(*)$, if W = vector field, $u \in T_pM$, $p \in M$.

$W(tu)$ in case of $M = \mathbb{R}^2$, replaced by

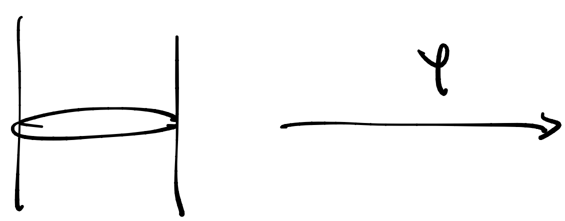
- $W(\alpha(t))$ where $\alpha(0) = p$, $\alpha'(0) = u$.
- $W(p)$ is now re-written as $W(\alpha(0)) \in T_pM$.

Then $\lim_{t \to 0} \dfrac{W(\alpha(t)) - W(\alpha(0))}{t}$ is well-defined in $\mathbb{R}^3$,

but NOT AS A VECTOR in T_pM!!

Looking for defn. only detect intrinsic geometry
(i.e. depends only on g, etc)

Example: M = cylinder



$\mathbb{R}^2/\sim$ where $x \sim y$ iff $x - y \in 2\pi\cdot\mathbb{Z} \times \{0\}$.

$\Psi(\cos u, \sin u, v) = (\tilde{u}, v)$ where $\tilde{u}$ = unique value in $[0, 2\pi)$ s.t. $e^{iu} = e^{i\tilde{u}}$.

And $\mathbb{R}^2/\sim$ has natural 1st f.f. $g_{ij} = \delta_{ij}$

while $(M, \tilde{g})$ has same 1st f.f. $\tilde{g}_{ij} = \delta_{ij}$

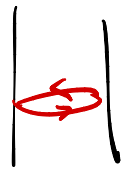
$\Rightarrow \Psi$ = isometry.

Consider $X : U \to M$ given by $X(u,v) = (\cos u, \sin u, v)$

Define $W = X_u$ (coordinate vector field)

$\alpha(0) = p = (1, 0, 0)$, $\alpha'(0) = X_u|_p = (-\sin u, \cos u, 0)|_p$

Choose $\alpha(t) = (\cos t, \sin t, 0)$ to be a differentiable curve on M.



Then $W(\alpha(t)) = (-\sin t, \cos t, 0)$

$$\Rightarrow \frac{d}{dt}\Big|_{t=0} W(\alpha(t)) = (-\cos t, -\sin t, 0)\Big|_{t=0} = (-1, 0, 0).$$

while

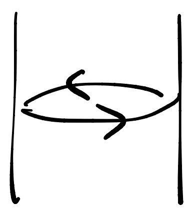
$$\begin{cases} d\varphi(W) = d\varphi(X_u) = \frac{\partial}{\partial u}(\varphi \circ X) = (1, 0). \\ \varphi \circ \alpha(t) = ([t], 0), \quad [t] \in 2\pi \cdot \mathbb{Z}. \end{cases}$$

s.t.

$$\frac{d}{dt}\Big|_{t=0} (d\varphi(W))(\varphi \circ \alpha(t)) = 0.$$

← Different.

Why different ??



$\frac{d}{dt} W(\alpha(t)) \neq 0$ Because α need acceleration to stay Inside M!!

The Real Acceleration $\Rightarrow \left(\frac{d}{dt}\Big|_{t=0} W(\alpha(t))\right)^T$

Defn: The covariant derivative of W at p, relative to $u \in T_pM$ is defined to be

$$D_u W\big|_p \triangleq \left(\frac{d}{dt}\Big|_{t=0} W(\alpha(t))\right)^T \in T_pM. \text{ where}$$

$\alpha: (-\varepsilon, \varepsilon) \to M$ is a differentiable curve w/

$$\alpha'(0) = u, \quad \alpha(0) = p.$$

(Some also denote as $\frac{DW}{dt}(0)$.)

<u>Rmk</u>: ① If $W(u,v) = f(u,v)\cdot X_u + h(u,v)\cdot X_v$

then $W\circ\alpha(t) = f(t)\cdot X_u + h(t)\cdot X_v$

$$\Rightarrow \frac{d}{dt} W\circ\alpha(t) = f' X_u + h' X_v \leftarrow \in T_{\alpha(t)}M$$

$$+ f X_u' + h X_v'$$

there might be some normal variation & many trouble.....

② $\because (T_pM)^\perp = \text{span}\{N\}$.

$$\therefore D_v W = \frac{d}{dt}\Big|_{t=0} W\circ\alpha(t) - \left\langle \frac{d}{dt}\Big|_{t=0} W\circ\alpha(t), N \right\rangle N.$$

OR

$$= D_v^{Euc} W - \langle D_v^{Euc} W, N\rangle N.$$

(directional derivative in $\mathbb{R}^3$)

<u>Q</u>: How to do computation on $X: \mathcal{U} \to M$??

let $\begin{cases} W = f X_u + h X_v \\ \alpha(t) = X(u(t), v(t)) \end{cases}$ be the local setting.

$$\Rightarrow \frac{d}{dt} W(\alpha(t)) = f\left(X_{uu}\cdot u' + X_{uv}\cdot v'\right) + f' X_u$$

$$+ h\left(X_{vu}\, u' + X_{vv}\, v'\right) + h' X_v.$$

where. $X_{uu} = \partial_u \partial_u X$ (as a vector in $\mathbb{R}^3$)

$= \Gamma^u_{uu} X_u + \Gamma^v_{uu} X_v + \mathrm{II}_{uu} N$

$$\Rightarrow \frac{DW}{dt} = f' X_u + h' X_v$$
$$+ f\left(\Gamma^u_{uu} X_u \cdot u' + \Gamma^v_{uu} X_v \cdot u' + \Gamma^u_{uv} X_u \cdot v' + \Gamma^v_{uv} X_v \cdot v' \right)$$
$$+ h\left(\Gamma^u_{uv} X_u u' + \Gamma^v_{uv} X_v u' + \Gamma^u_{vv} X_u v' + \Gamma^v_{vv} X_v v' \right)$$

* Is invariant under local isometry

$\Rightarrow$ intrinsic in natural.

Observe also, the defn above only requires W on $\alpha(t)$.

Defn: A vector field W along a parametrized curve $\alpha: I \to M$ if $\forall t \in I$, $W(t) \in T_{\alpha(t)}M$.

Likewise W is differentiable if $W = a(t) X_u + b(t) X_v$ for some differentiable fcn $a(t), b(t)$ on I.

$\to$ Define $\left.\frac{DW}{dt}\right|_{t=0}$ similarly by $\left(\left.\frac{d}{dt}\right|_{t=0} W(t)\right)^T$.

Denoted as $D_{\alpha'}W$.

Special case: Along $\alpha : I \to M$,

taking $W(t) = \alpha'(t)$ (= velocity)

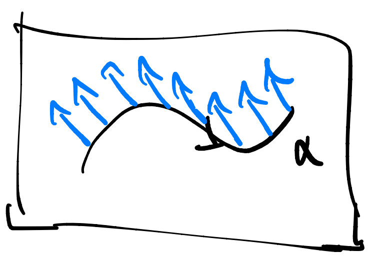
$$D_{\alpha'}W\Big|_{t=0} = \left(\frac{d}{dt}\Big|_{t=0} W(t)\right)^T = \left(\frac{d}{dt}\Big|_{t=0} \alpha'(t)\right)^T$$

$$= \text{tangential acceleration.}$$

(acceleration seen on M).

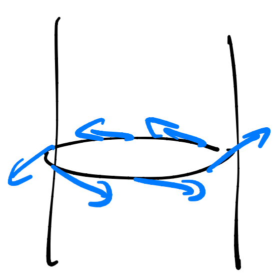
Defn: A vector field W along $\alpha : I \to M$ is said to be parallel if $D_{\alpha'}W \equiv 0 \;\; \forall t \in I$.

eg:

① 

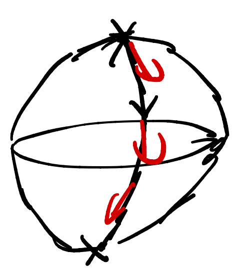
M = plane

W = parallel along α.

② 

M = cylinder

W = parallel along α

③ 

$M = S^2$

W = parallel along great circle.

prop: If $W(t), V(t)$ = parallel vector field along $\alpha : I \to M$, then $\langle W(t), V(t) \rangle = \langle W(0), V(0) \rangle$

pf: $\frac{d}{dt} \langle V, W \rangle = \langle \frac{d}{dt} V, W \rangle + \langle V, \frac{d}{dt} W \rangle$

$\because \frac{DV}{dt} \in T_pM$. $\to$ $= \langle \frac{DV}{dt}, W \rangle + \langle V, \frac{DW}{dt} \rangle$

$= 0$.

$\Rightarrow$ constant in $t \in I$ $\blacksquare$.

(Putting $V = W \Rightarrow \|V\| \equiv$ constant along $\alpha : I \to M$)

Discussion: Given $\alpha : I \to M$, a regular parametrized curve

A vector field W is parallel if $D_{\alpha'} W = 0$

locally, $W = a(t) X_u + b(t) X_v$.

$D_{\alpha'} W = 0 \iff$

unknown

$$\begin{cases} 0 = a'(t) + \Gamma_{11}^1 a(t) \cdot u'(t) + \Gamma_{12}^1 a(t) v'(t) \\ \quad + \Gamma_{12}^1 b(t) \cdot u'(t) + \Gamma_{22}^1 b(t) \cdot v'(t) \\ 0 = b'(t) + \cdots \end{cases}$$

given

This is a system of 1st order linear ODE

$\Rightarrow$ Existence & unique. of sol.

$\Downarrow$

prop: Given $\alpha: I \to M$, a regular parametrized curve, $t_0 \in I$, $\alpha(t_0) \in M$, $W_0 \in T_{\alpha(t_0)}M$.

$\exists!$ parallel vector field $W(t)$ with $W(t_0) = W_0$.

☆☆ "The vector $W(t_1)$, $t_1 \in I$ ($\neq t_0$) is called the parallel transport of W_0 along α at $t=t_1$."

(Some ppl denote as $P_\alpha W_0$.)

Defn: (geodesic) A non-constant, parametrized curve $\gamma: I \to M$ is said to be a geodesic if $D_{\gamma'}\gamma' \equiv 0$ on I.

(locally, $= \left(\frac{d}{dt}\alpha'\right)^T = 0$, $\forall t \in I$)

Previous prop $\Rightarrow$ geodesic has constant speed

Compare with discussion in principle curv.:

Given a regular curve $\alpha : I \to M$, parametrized by arc-length. Given M an orientation N.

Then $\alpha'' = k_n N + k_g n \in \mathbb{R}^3$

(k_n: normal curv.; k_g: geodesic curv.)

$\Rightarrow D_{\alpha'}\alpha' \triangleq (\alpha'')^T = k_g \cdot n$

- If, (in addition) α = geodesic, then $k_g \equiv 0$.

geodesic $\Rightarrow$ arc-length parametrization (up to scaling)

geodesic $\Rightarrow$ $k_g \equiv 0$.

Conversely, if $|\alpha''| \equiv 1$, $k_g \equiv 0$

then $D_{\alpha'}\alpha' \equiv 0$ (geodesic).

Geometrically, no acceleration $\Rightarrow$ "Straight line".

$\therefore$ geodesic "=" straight line (meaning??)